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グラスマン束の次数公式

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1. INTRODUCTION

The purpose of our work is to give degree formulae for Grassmann bundles. This article is a summary of a joint paper [4] with Tomohide Terasoma.

Let X be a projective variety of dimension n over a field of arbitrary characteristic, let \mathcal{E} be a vector bundle of rank r on X , let $\mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of \mathcal{E} on X with projection $\pi : \mathbb{G}_X(d, \mathcal{E}) \rightarrow X$, and let $\pi^*\mathcal{E} \rightarrow \mathcal{Q}$ be the universal quotient bundle of rank d . Set $\theta := c_1(\mathcal{Q})$, the first Chern class of \mathcal{Q} , whose determinant bundle, $\det \mathcal{Q}$, is isomorphic to the pull-back of the tautological line bundle of $\mathbb{P}_X(\wedge^d \mathcal{E})$ by the (relative) Plücker embedding over X . In this article we call θ the *Plücker class* of $\mathbb{G}_X(d, \mathcal{E})$. The theme discussed here is how to calculate the self-intersection number of the Plücker class, $\int_{\mathbb{G}_X(d, \mathcal{E})} \theta^N$, which is the degree of $\mathbb{G}_X(d, \mathcal{E})$ embedded in the projective space $\mathbb{P}(H^0(X, \wedge^d \mathcal{E}))$ via the Plücker embedding if $\wedge^d \mathcal{E}$ is very ample, where $N := \dim \mathbb{G}_X(d, \mathcal{E}) = d(r - d) + n$.

The result is

Theorem 1.1. *Let θ be the Plücker class of $\mathbb{G}_X(d, \mathcal{E})$. Then*

(1)

$$\int_{\mathbb{G}_X(d, \mathcal{E})} \theta^N = N! \sum_{|k|=n} \frac{\prod_{1 \leq i < j \leq d} (k_i - k_j - i + j)}{\prod_{1 \leq i \leq d} (r + k_i - i)!} \int_X \prod_{1 \leq i \leq d} s_{k_i}(\mathcal{E}),$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}_{\geq 0}^d$ with $|k| := \sum_i k_i$, and $s_i(\mathcal{E})$ is the i -th Segre class of \mathcal{E} .

(2)

$$\int_{\mathbb{G}_X(d, \mathcal{E})} \theta^N = N! \sum_{|\lambda|=n} \frac{\prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j - i + j)}{\prod_{1 \leq i \leq d} (r + \lambda_i - i)!} \int_X \Delta_\lambda(s(\mathcal{E})),$$

where $\Delta_\lambda(s(\mathcal{E}))$ is the Schur polynomial of \mathcal{E} for a partition $\lambda = (\lambda_1, \dots, \lambda_d)$.

In fact, we give two formulae for $\pi_* \text{ch}(\det \mathcal{Q})$, the push-forward of the Chern character of $\det \mathcal{Q}$ by π , explicitly (Theorem 2.1), under the assumption that X is a scheme of finite type over a field k : The above result is a direct consequence of those formulae.

The Segre classes $s_i(\mathcal{E})$ here are the ones satisfying $s(\mathcal{E}, t)c(\mathcal{E}, -t) = 1$ as in [1], [5], where $s(\mathcal{E}, t)$ and $c(\mathcal{E}, t)$ are respectively the Segre series and the Chern polynomial of \mathcal{E} in t . Note that our Segre class $s_i(\mathcal{E})$ differs by the sign $(-1)^i$ from the one in [2].

Theorem 1.1 with $n = 0$ yields the degree formula of Grassmann varieties, as follows:

Corollary 1.2 ([2, Example 14.7.11 (iii)]). *The degree of the Grassmann variety $\mathbb{G}(d, r)$ of codimension d subspaces of an r -dimensional vector space with respect to the Plücker embedding is given by*

$$\deg \mathbb{G}(d, r) = \frac{(d(r-d))! \prod_{1 \leq k \leq d-1} k!}{\prod_{1 \leq k \leq d} (r-k)!}.$$

2. MAIN RESULTS

Theorem 1.1 follows from more general results, as follows: Setting $m! := \Gamma(m+1)$ for $m \in \mathbb{Z}$, one has $1/m! = 0$ if $m < 0$. To simplify the notation, for a finite set of integers $\{a_i\}_{0 \leq i \leq d-1}$, set

$$\{a_i\}! := \prod_l a_l!, \quad \Delta(a_i) := \prod_{i < j} (a_i - a_j).$$

Theorem 2.1. *Assume that X is a scheme of finite type over a field k . Let $\mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of a vector bundle \mathcal{E} of rank r on X with projection $\pi : \mathbb{G}_X(d, \mathcal{E}) \rightarrow X$, let $\pi^* \mathcal{E} \rightarrow \mathcal{Q}$ be the universal quotient bundle of rank d , and let $\text{ch}(\det \mathcal{Q})$ be the Chern character of $\det \mathcal{Q}$. Denote by $\pi_* : A^*(\mathbb{G}_X(d, \mathcal{E})) \otimes \mathbb{Q} \rightarrow A^{*-d(r-d)}(X) \otimes \mathbb{Q}$ is the push-forward by π . Then*

(1)

$$\pi_* \text{ch}(\det \mathcal{Q}) = \sum_k \frac{\Delta(k_i - i)}{\{r + k_i - i\}!} \prod_{1 \leq l \leq d} s_{k_l}(\mathcal{E}),$$

where $k = (k_1, \dots, k_d) \in \mathbb{Z}_{\geq 0}^d$, and $s_i(\mathcal{E})$ is the i -th Segre class of \mathcal{E} .

(2)

$$\pi_* \operatorname{ch}(\det \mathcal{Q}) = \sum_{\lambda} \frac{\Delta(\lambda_i - i)}{\{r + \lambda_i - i\}!} \Delta_{\lambda}(s(\mathcal{E})),$$

where $\Delta_{\lambda}(s(\mathcal{E})) := \det[s_{\lambda_i + j - i}(\mathcal{E})]_{1 \leq i, j \leq d}$ is the Schur polynomial of \mathcal{E} for a partition $\lambda = (\lambda_1, \dots, \lambda_d)$.

3. (SKETCH OF)² PROOF

Let X be a scheme of finite type over a field k , and let \mathcal{E} be a vector bundle of rank r on X . Denote by $\mathbb{F}_X^d(\mathcal{E})$ the partial flag bundle of \mathcal{E} on X , parametrising flags of subbundles of corank 1 up to d in \mathcal{E} . Then it is easily shown that the projection $p : \mathbb{F}_X^d(\mathcal{E}) \rightarrow X$ decomposes as a successive composition of projective space bundles, $\mathbb{P}(\mathcal{E}_i)/\mathbb{P}(\mathcal{E}_{i-1})$ ($i \geq 1$):

$$p : \mathbb{F}_X^d(\mathcal{E}) = \mathbb{P}(\mathcal{E}_{d-1}) \rightarrow \mathbb{P}(\mathcal{E}_{d-2}) \rightarrow \dots \rightarrow \mathbb{P}(\mathcal{E}_1) \rightarrow \mathbb{P}(\mathcal{E}_0) \rightarrow X,$$

where $\mathcal{E}_0 := \mathcal{E}$, and \mathcal{E}_{i+1} is the kernel of the canonical surjection from the pull-back of \mathcal{E}_i to $\mathbb{P}(\mathcal{E}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)$ with $\operatorname{rk} \mathcal{E}_i = r - i$ ($i \geq 0$): In fact, $\mathbb{P}(\mathcal{E}_i) \simeq \mathbb{F}_X^{i+1}(\mathcal{E})$ ($1 \leq i \leq d-1$). Set $\xi_i := c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1))$. Then, the intersection ring of $A^*(\mathbb{F}_X^d(\mathcal{E}))$ is given as follows:

$$\begin{aligned} A^*(\mathbb{F}_X^d(\mathcal{E})) &= \frac{A^*(X)[\xi_0, \xi_1, \dots, \xi_{d-1}]}{(P_0(\xi_0), P_1(\xi_1), \dots, P_{d-1}(\xi_{d-1}))} \\ (3.1) \quad &= \bigoplus_{\substack{0 \leq i_l \leq r-l-1 \\ (0 \leq l \leq d-1)}} A^*(X) \overline{\xi_0}^{i_0} \overline{\xi_1}^{i_1} \dots \overline{\xi_{d-1}}^{i_{d-1}}, \end{aligned}$$

where $P_i(\xi_i) := \xi_i^{r-i} - c_1(\mathcal{E}_i) \xi_i^{r-i-1} + \dots + (-1)^{r-i} c_{r-i}(\mathcal{E}_i) \in A^*(\mathbb{P}(\mathcal{E}_i))[\xi_i]$, and the symbol of pull-back to $\mathbb{F}_X^d(\mathcal{E})$ is omitted. Denote by $p_* : A^*(\mathbb{F}_X^d(\mathcal{E})) \rightarrow A^{*-c}(X)$ the push-forward by p , where $c := \sum_{0 \leq i \leq d-1} (r-i-1)$, the relative dimension of $\mathbb{F}_X^d(\mathcal{E})/X$. Then, for $\alpha = \sum \alpha_{i_0 i_1 \dots i_{d-1}} \overline{\xi_0}^{i_0} \overline{\xi_1}^{i_1} \dots \overline{\xi_{d-1}}^{i_{d-1}}$ in $A^*(\mathbb{F}_X^d(\mathcal{E}))$ ($\alpha_{i_0 i_1 \dots i_{d-1}} \in A^*(X)$) with respect to the decomposition in (3.1), one has

$$(3.2) \quad p_* \alpha = \alpha_{r-1, r-2, \dots, r-d}.$$

Indeed, $\sum_l i_l \geq c$ if and only if $i_l = r - l - 1$ for each l .

Let $G := \mathbb{G}_X(d, \mathcal{E})$ be the Grassmann bundle of corank d subbundles of \mathcal{E} on X , and let $\pi^* \mathcal{E} \rightarrow \mathcal{Q}$ be the universal quotient bundle of rank d . Consider the flag bundle $\mathbb{F}_G^{d-1}(\mathcal{Q})$ of \mathcal{Q} on G , parametrising flags of subbundles of corank 1 up to $d-1$ in \mathcal{Q} . Then, as in the case of $\mathbb{F}_X^d(\mathcal{E})$,

the projection $\mathbb{F}_G^{d-1}(\mathcal{Q}) \rightarrow G$ decomposes as a successive composition of projective space bundles $\mathbb{P}(\mathcal{Q}_{i+1})/\mathbb{P}(\mathcal{Q}_i)$ ($i \geq 1$):

$$q : \mathbb{F}_G^{d-1}(\mathcal{Q}) = \mathbb{P}(\mathcal{Q}_{d-2}) \rightarrow \mathbb{P}(\mathcal{Q}_{d-2}) \rightarrow \cdots \rightarrow \mathbb{P}(\mathcal{Q}_1) \rightarrow \mathbb{P}(\mathcal{Q}_0) \rightarrow G,$$

where $\mathcal{Q}_0 := \mathcal{Q}$, and \mathcal{Q}_{i+1} is the kernel of the canonical surjection from the pull-back of \mathcal{Q}_i to $\mathbb{P}(\mathcal{Q}_i)$, to the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1)$ with $\text{rk } \mathcal{Q}_i = d - i$ ($i \geq 0$): In fact, $\mathbb{P}(\mathcal{Q}_i) \simeq \mathbb{F}_G^{i+1}(\mathcal{Q})$ ($1 \leq i \leq d - 2$).

It follows from the construction of vector bundles \mathcal{E}_i that \mathcal{E}_d is a corank d subbundle of $p^*\mathcal{E}$ on $\mathbb{F}_X^d(\mathcal{E})$, which induces a morphism, $r : \mathbb{F}_X^d(\mathcal{E}) \rightarrow G$ by the universal property of the Grassmann bundle G . Then it turns out that $\mathbb{F}_G^{d-1}(\mathcal{Q})$ is naturally isomorphic to $\mathbb{F}_X^d(\mathcal{E})$ over G via r , as is easily verified by using the universal property of flag bundles (see [5, §6], [7, §§0–1]): We identify them via the natural isomorphism $\mathbb{F}_G^{d-1}(\mathcal{Q}) \simeq \mathbb{F}_X^d(\mathcal{E})$. Under this identification, it follows that

$$\xi_i = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_i)}(1)) = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{Q}_i)}(1))$$

in $A^*(\mathbb{F}_X^d(\mathcal{E})) = A^*(\mathbb{F}_G^{d-1}(\mathcal{Q}))$, where the symbol of pull-back to $\mathbb{F}_X^d(\mathcal{E}) = \mathbb{F}_G^{d-1}(\mathcal{Q})$ is omitted, as before.

For the Plücker class $\theta = c_1(\mathcal{Q})$, one has

Lemma 3.1. (1) $\theta^N = q_*(\xi_0^{d-1}\xi_1^{d-2}\cdots\xi_{d-2}q^*\theta^N)$ in $A^*(G)$.
 (2) $q^*\theta = \xi_0 + \cdots + \xi_{d-1}$ in $A^*(\mathbb{F}_X^d(\mathcal{E})) = A^*(\mathbb{F}_G^{d-1}(\mathcal{Q}))$.

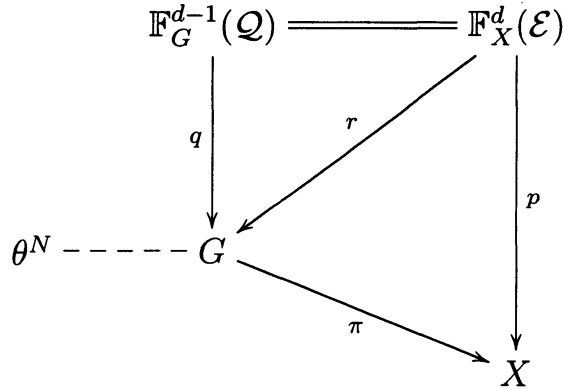


FIGURE 1

It follows from Lemma 3.1, the commutativity $p = \pi \circ q$ and (3.2) that

$$\begin{aligned}
 \pi_*(\theta^N) &= \pi_* q_* (\xi_0^{d-1} \xi_1^{d-2} \cdots \xi_{d-2} q^* \theta^N) = \pi_* q_* \left(\prod_{i=0}^{d-1} \xi_i^{d-1-i} \left(\sum_{i=0}^{d-1} \xi_i \right)^N \right) \\
 (3.3) \quad &= p_* \left(\prod_{i=0}^{d-1} \xi_i^{d-1-i} \left(\sum_{i=0}^{d-1} \xi_i \right)^N \right) \\
 &= \text{coeff}_{\overline{\xi_0}, \dots, \overline{\xi_{d-1}}} \left(\prod_{i=0}^{d-1} \xi_i^{d-1-i} \left(\sum_{i=0}^{d-1} \xi_i \right)^N ; r-1, \dots, r-d \right),
 \end{aligned}$$

where $\text{coeff}_{\overline{\xi_0}, \dots, \overline{\xi_{d-1}}}(\cdots ; r-1, \dots, r-d)$ denotes the coefficient of \cdots in $\overline{\xi_0}^{r-1} \overline{\xi_1}^{r-2} \cdots \overline{\xi_{d-1}}^{r-d}$.

Now one can show that

Lemma 3.2.

$$\text{coeff}_{\overline{\xi_i}}(\xi_i^{p_i}; r-i-1) = \text{const}_{t_i}(t_i^{-p_i+r-i-1} s(\mathcal{E}_i, t_i)),$$

where $\text{const}_{t_i}(\cdots)$ the constant term in the Laurent expansion of \cdots in t_i .

Applying Lemma 3.2 repeatedly, one obtains

Lemma 3.3.

$$\begin{aligned}
 \text{coeff}_{\overline{\xi_0}, \dots, \overline{\xi_{d-1}}}(\xi_0^{p_0} \cdots \xi_{d-1}^{p_{d-1}}; r-1, \dots, r-d) \\
 = \text{const}_{\underline{t}} \left(\Delta(t_0, \dots, t_{d-1}) \prod_{i=0}^{d-1} t_i^{-p_i+r-d} s(\mathcal{E}_0, t_i) \right),
 \end{aligned}$$

where $\underline{t} := (t_0, \dots, t_{d-1})$, and $\Delta(t_0, \dots, t_{d-1}) := \prod_{0 \leq i < j \leq d-1} (t_i - t_j)$ the Vandermonde polynomial of (t_0, \dots, t_{d-1}) .

By virtue of (3.3) and Lemma 3.3, one can show

Proposition 3.4. For a non-negative integer N ,

$$\pi_* \theta^N = \text{const}_{\underline{t}}(P_N(\underline{t})),$$

where $\pi_* : A^*(\mathbb{G}_X(d, \mathcal{E})) \rightarrow A^{*-d(r-d)}(X)$ is the push-forward by π , $s(\mathcal{E}, t)$ is the Segre series of \mathcal{E} in t , and

$$P_N(\underline{t}) := \Delta(\underline{t}) \left(\sum_{i=0}^{d-1} \frac{1}{t_i} \right)^N \prod_{i=0}^{d-1} t_i^{-(d-1-i)+r-d} s(\mathcal{E}_0, t_i).$$

Now, to prove Theorem 2.1 (1), just expand the Laurent series $P_N(\underline{t})$ by the multinomial theorem with the following

Lemma 3.5 ([2, Example A.9.3]).

$$\det \left[\frac{1}{(x_i + j)!} \right]_{0 \leq i, j \leq d-1} = \frac{\Delta(x_i)}{\{x_i + d - 1\}!}.$$

For Theorem 2.1 (2), we have two proofs, where we use a consequence of Cauchy identity [6, Chapter I, (4.3)] and Jacobi-Trudi identity [2, Lemma A.9.3], as follows:

Lemma 3.6.

$$\prod_{i=0}^{d-1} s(\mathcal{E}, t_i) = \sum_{\lambda \geq 0} \Delta_\lambda(s(\mathcal{E})) s_\lambda(\underline{t}).$$

One of our proofs is obtained just by expanding $P_N(\underline{t})$, similarly to the proof of Theorem 2.1 (1). For the other, we establish a formula of Kadell type for confluent Selberg integral, due to Terasoma, as follows (Cf. [3]):

Proposition 3.7. *Set*

$$W_{\text{exp}}(x, \underline{t}) := \prod_{i=0}^{d-1} t_i^{x-1} \prod_{i=0}^{d-1} \exp(-t_i) \prod_{i < j} (t_i - t_j)^2,$$

$$I_{\text{conf}}(\lambda, x) := \int_{[0, +\infty)^d} s_\lambda(\underline{t}) W_{\text{exp}}(x, \underline{t}) d\underline{t}.$$

Then

$$I_{\text{conf}}(\lambda, x) = d! \Delta(\lambda_i - i) \Gamma\{x + d - i + \lambda_i\},$$

for a real number $x > 0$, where $\underline{t} := (t_0, \dots, t_{d-1})$ and $d\underline{t} := dt_0 \cdots dt_{d-1}$.

Remark 3.8. Symmetrising the Laurent series $P_N(\underline{t})$ with respect to the variables \underline{t} , one sees that $\text{const}_{\underline{t}}(P_N(\underline{t}))$ is equal to the constant term of the Laurent series,

$$P_N^s(\underline{t}) := \frac{(-1)^{\frac{d(d-1)}{2}}}{d!} \prod_{0 \leq i < j \leq d-1} \left(\frac{1}{t_i} - \frac{1}{t_j} \right)^2 \left(\sum_{i=0}^{d-1} \frac{1}{t_i} \right)^N \prod_{i=0}^{d-1} t_i^{r-1} s(\mathcal{E}, t_i).$$

Roughly speaking, to obtain the constant term of $P_N(\underline{t})$, we calculate the residue of $P_N^s(t_0^{-1}, \dots, t_{d-1}^{-1})(t_0 \cdots t_{d-1})^{-1}$ by using Proposition 3.7 (see Remark 3.8): Indeed, we have $\text{const}_{\underline{t}}(P_N(\underline{t})) = \text{const}_{\underline{t}}(P_N^s(t_0^{-1}, \dots, t_{d-1}^{-1}))$.

4. EXAMPLE

Example 4.1. $\deg \mathbb{G}_{\mathbb{P}^4}(2, T_{\mathbb{P}^4}) = 5040$. This number is exactly equal to the factorial of 7 (pointed out by Agaoka): $5040 = 7!$. I guess this would be nothing but a coincidence without rationale (what do you think?).

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